

Time of Arrival and Angle of Arrival Statistics for Distant Circular Scattering Model

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Abstrak

Model hamburan umum adalah proses hamburan lokal diasumsikan bahwa stasiun bergerak terletak di dalam wilayah hamburan. Untuk mempelajari model hamburan jauh di lingkungan propagasi pinggiran kota atau berbukit, sebuah modus hamburan melingkar jauh secara statistik berdasarkan geometri di lingkungan sel makro diusulkan. Ekspresi bentuk tertutup dari fungsi kepadatan probabilitas gabungan dari waktu kedatangan/sudut kedatangan, kepadatan fungsi probabilitas marjinal sudut kedatangan dan sudut keberangkatan, kepadatan fungsi probabilitas marjinal waktu kedatangan diturunkan. fungsi kepadatan probabilitas ini memberikan wawasan ke dalam sifat dari model hamburan saluran spasial jauh.

Kata kunci: model hamburan jauh, sudut datang, sudut keberangkatan, waktu datang

Abstract

General scattering model is local scattering process assumed that the mobile station is located inside the scattering region, in order to study distant scattering model in suburban or hilly propagation environment, a geometrically based statistical distant circular scattering mode in macrocell environment was proposed, the closed-form expressions of the joint probability density function of the time of arrival /the angle of arrival, the marginal probability density function of the angle of arrival and the angle of departure, the marginal probability density function of the time of arrival were derived, this probability density function's provided insight into the properties of the spatial distant scattering channel model.

Keywords: distant scattering model, the angle of arrival, the angle of departure, the time of arrival

1. Introduction

Recently the use of smart antennas and beamforming technique has motivated to research the model for the spatial characteristics of the cellular mobile channel, various scattering models and related issues can be found in [1-10], the ring model [2], the scatterers are uniformly distributed on a ring which is centered about the mobile station (MS). Circular scattering model [4,5], the density of scatterers within a circular region about MS has been assumed to be uniform, this model was assumed to be most valid in macrocell environments where the antenna heights of the base station (BS) are relatively high without signal scattering from location near BS. Elliptical scattering model [6], where the scatterers are uniformly distributed within an elliptical region with foci at BS and MS respectively, this model was assumed valid in micro or picocell types of environments where the angular spread tends to be high and the delay spread tends to be low. References [7-10] gave some scattering model of non-uniform probability density function, such as Gaussian density scatter model [7], it assumed that the mobile station is surrounded by the scatterers of Gaussian distribution; the model was used in the study of diversity antennas and smart antennas. Rayleigh and exponential distributions model [8], analysis shows that the Rayleigh distribution scatterer model can predict the outdoor environment, while the exponential distribution scatterer model is suitable for the indoor environment. The hyperbolic scatterers distribution model [9], it has been shown that the hyperbolic the angle of arrival (AOA) probability density function is more effective than the Gaussian AOA probability density function when scatterers radius $R \leq 3.3$ km. A 3-D model is proposed in [10], which can be considered applicable to micro- and picocell environments.

Nevertheless, Most of the existing geometric channel models take into account only the local scattering, the mobile station is located inside the scattering region, and few available models define the shape and distribution of distance scattering. Distance scattering is the scattering process which results from the dominant distant scattering structures far from both the BS and the MS, this type of scattering can occur in hilly and suburban areas due to large scattering structures such as mountains and high building clusters, which have a significant influence on the mobile channel^[11]. The contributions of this paper is to study a distant circular scattering model, and derive the joint and marginal probability density functions of the angle of arrival and the time of arrival (TOA) in closed form for distant scattering in macrocell environment, which are required to test adaptive array algorithms for cellular applications..

2. Distant Circular Scattering Model

Figure1 shows the proposed distant circular scattering geometry model in macrocell environment, it is assumed that the signals received at BS have interacted with only a single scatterer in the channel, the MS is located on the x -axis with the origin at BS, D denotes the distance between MS and BS, a point P is the centre of the distant scattering circular with a radius R and is denoted by the polar coordinates (d_p, θ_p) and (d_r, θ_r) with respect to the polar coordinates with the origin at BS and MS respectively, the model assumes that the scatterers are uniformly distributed within the circular, a random scatterer position $S(x_s, y_s)$ can denoted by (d, θ_d) and (r, θ_r) with respect to the polar coordinates with the origin at BS and MS respectively, it is assumed that the distant scattering circular is located inside the ellipse with foci at BS and MS, its semimajor axis a_m and semiminor axis b_m values are given by:

$$a_m = c\tau_m / 2 \quad (1)$$

$$b_m = \sqrt{c^2\tau_m^2 - D^2} / 2 \quad (2)$$

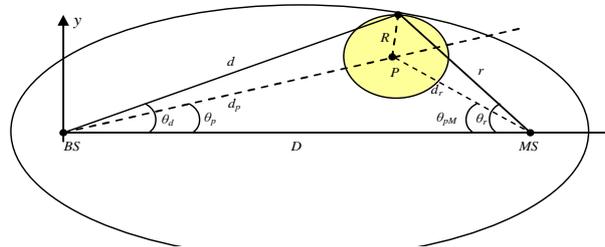


Figure 1. Distant scattering model

Where τ_m is the maximum delay associated with scatterers within the ellipse. As shown in Figure1, scatterers were assumed to be uniformly spread over the distant circular, the scatter density function in rectangular coordinates can be written as:

$$f_{x_s, y_s}(x_s, y_s) = \begin{cases} \frac{1}{\pi R^2} & |(x_s, y_s)| \leq R \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The total path propagation delay is given by

$$\tau = d + r \quad (4)$$

Looking at the triangle $\Delta(MS, BS, S)$ as shown in Figure1, the following equation can be deduced:

$$\tau - d = \sqrt{d^2 + D^2 - 2dD \cos(\theta_d)} \quad (5)$$

Squaring both sides of (5) and solving for d results in

$$d = \frac{D^2 - \tau^2 c^2}{2(D \cos \theta_d - \tau)} \quad (6)$$

Therefore, the location of the scatterer in the rectangular coordinates is given by;

$$x_s = d \cos \theta_d = \frac{(D^2 - \tau^2 c^2) \cos \theta_d}{2(D \cos \theta_d - \tau)} \quad (7)$$

$$y_s = d \sin \theta_d = \frac{(D^2 - \tau^2 c^2) \sin \theta_d}{2(D \cos \theta_d - \tau)} \quad (8)$$

2.1. Joint TOA/AOA pdf at BS

Once the scatterer distribution $f_{x_s, y_s}(x_s, y_s)$ and the relations between (x_s, y_s) and (τ, θ_d) are known, the joint TOA/AOA pdf $f_{\tau, \theta_d}(\tau, \theta_d)$ can be obtained by a Jacobian transformation between (x_s, y_s) and (τ, θ_d) :

$$\begin{aligned} f_{\tau, \theta_d}(\tau, \theta_d) &= f_{x_s, y_s}(x_s, y_s) \left| \frac{\frac{\partial x_s}{\partial \tau} \quad \frac{\partial y_s}{\partial \tau}}{\frac{\partial x_s}{\partial \theta_d} \quad \frac{\partial y_s}{\partial \theta_d}} \right| = f_{x_s, y_s}(x_s, y_s) \cdot \frac{(D^2 - \tau^2 c^2)(D^2 + \tau^2 c^2 - 2D\tau \cos \theta_d)c}{4(D \cos \theta_d - \tau)^3} \\ &= \frac{(D^2 - \tau^2 c^2)(D^2 + \tau^2 c^2 - 2D\tau \cos \theta_d)c}{4\pi R^2 (D \cos \theta_d - \tau)^3} \end{aligned} \quad (9)$$

2.2. AOA pdf at BS

The AOA pdf could be found by integrate the polar coordinate system representation of the scatterer density function $f_{d, \theta_d}(d, \theta_d)$ with respect to d over the range d_1 to d_2 , where d_1 and d_2 are two pairs of roots for the equations defining the distant scattering discs, in polar coordinates:

$$d^2 + d_p^2 - 2dd_p \cos(\theta_d - \theta_p) \in [0, R^2] \quad (10)$$

Equate (10) to its upper limit of R^2 producing the following two roots

$$d_1, d_2 = \{d_p \cos(\theta_d - \theta_p) \pm \sqrt{R^2 - d_p^2 \sin^2(\theta_d - \theta_p)}\} \quad (11)$$

When the relations between (x_s, y_s) and (d, θ_d) are known, the joint pdf $f_{d, \theta_d}(d, \theta_d)$ can be obtained by another Jacobian transformation between (x_s, y_s) and (d, θ_d) :

$$f_{d, \theta_d}(d, \theta_d) = f_{x_s, y_s}(x_s, y_s) \left| \frac{\frac{\partial x_s}{\partial d} \quad \frac{\partial y_s}{\partial d}}{\frac{\partial x_s}{\partial \theta_d} \quad \frac{\partial y_s}{\partial \theta_d}} \right|_{\substack{x_s = d \cos \theta_d \\ y_s = d \sin \theta_d}} = df_{x_s, y_s}(d \cos \theta_d, d \sin \theta_d) \quad (12)$$

Therefore the AOA pdf at BS equals

$$f_{\theta_d}(\theta_d) = \int_{d_1}^{d_2} f_{d, \theta_d}(d, \theta_d) dd = \int_{d_1}^{d_2} d \cdot \frac{1}{\pi R^2} dd = \frac{d_2^2 - d_1^2}{2\pi R^2}$$

$$= \begin{cases} \frac{2d_p \cos(\theta_d - \theta_p) \sqrt{R^2 - d_p^2 \sin^2(\theta_d - \theta_p)}}{\pi R^2} & \theta_p - \arcsin \frac{R}{d_p} \leq \theta_d \leq \theta_p + \arcsin \frac{R}{d_p} \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

2.3. TOA pdf at BS

To identify the support region of θ at a specific τ , there exists a τ -constant spatial ellipse focusing at the base station's and the mobile station's spatial locations, any propagation path must bounce off a scatterer lying on this ellipse's rim, this elliptical rim intersects with the circle (within which the scatterers lie) at two points at most^[12], that is θ satisfying the following relation for a specific time delay τ :

$$\left(\frac{D^2 - \tau^2 c^2}{2(D \cos \theta_d - \tau c)} \right)^2 + d_p^2 - \frac{(D^2 - \tau^2 c^2) d_p}{(D \cos \theta_d - \tau c)} \cos(\theta_d - \theta_p) = R^2 \quad (14)$$

In order to simplify calculations, we use another method, in rectangular coordinates the intersection points (x, y) satisfy the following both equations:

$$\begin{cases} (x - x_0)^2 + (y - y_0)^2 = R^2 \\ \frac{(x - D/2)^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases} \quad (15)$$

Where $x_0 = d_p \cos \theta_p$, $y_0 = d_p \sin \theta_p$, $a = \tau c / 2$, $b = \sqrt{c^2 \tau^2 - D^2} / 2$, after some elementary but rather tedious calculation, equation (15) can be expressed as:

$$a_i x^4 + b_i x^3 + c_i x^2 + d_i x + e_i = 0 \quad (16)$$

where:

$$\begin{aligned} a_i &= -\frac{2b^2}{a^2} + 1 + \frac{b^4}{a^4} \\ b_i &= \frac{2Db^2}{a^2} - 4x_0 + \frac{4b^2 x_0}{a^2} - \frac{2Db^4}{a^4} \\ c_i &= 2y_0^2 - 2R^2 - \frac{2b^2 x_0^2}{a^2} + \frac{2y_0^2 b^2}{a^2} + 6x_0^2 + 2b^2 - \frac{4Db^2 x_0}{a^2} - \frac{2b^4}{a^2} + \frac{3D^2 b^4}{2a^4} + \frac{2R^2 b^2}{a^2} - \frac{b^2 D^2}{2a^2} \\ d_i &= -4x_0^3 + \frac{2Db^2 x_0^2}{a^2} - 4b^2 x_0 - \frac{2y_0^2 Db^2}{a^2} + \frac{D^2 b^2 x_0}{a^2} - 4y_0^2 x_0 + 4x_0 R^2 + \frac{2Db^4}{a^2} - \frac{b^4 D^3}{2a^4} - \frac{2DR^2 b^2}{a^2} \\ e_i &= -2y_0^2 b^2 + \frac{y_0^2 b^2 D^2}{2a^2} + b^4 - \frac{D^2 b^4}{2a^2} - 2b^2 R^2 + \frac{D^4 b^4}{16a^4} + \frac{R^2 b^2 D^2}{2a^2} + y_0^4 - 2R^2 y_0^2 + R^4 + 2x_0^2 y_0^2 \\ &\quad - 2x_0^2 R^2 + x_0^4 + 2b^2 x_0^2 - \frac{b^2 D^2 x_0^2}{2a^2} \end{aligned}$$

Equation (16) is a quartic equation, the quartic can be solved by means of a method discovered by Lodovico Ferrari^[14], and its two roots satisfying the following relationships are the solution to the equation (15), we express the roots as x_1, x_2 and $x_1 \leq x_2$:

$$d_p \cos \theta_p - R \leq x_1 \leq x_2 \leq d_p \cos \theta_p + R \quad (17)$$

Then in polar coordinates, the angles of the two intersect points are

$$\theta_{d2} = (\arctan \sqrt{b^2 - \frac{b^2(x_1 - D/2)^2}{a^2}}) / x_1 \quad (18)$$

$$\theta_{d1} = (\arctan \sqrt{b^2 - \frac{b^2(x_2 - D/2)^2}{a^2}}) / x_2 \quad (19)$$

In addition, the minim and maximum AOA for distant scattering are given by:

$$\theta_{\min 1} = \theta_p - \arcsin \frac{R}{d_p} \quad (20)$$

$$\theta_{\max 1} = \theta_p + \arcsin \frac{R}{d_p} \quad (21)$$

using $d = \frac{D^2 - \tau^2 c^2}{2(D \cos \theta_d - \tau)}$, $\theta_d = \theta_{\max}$, and $d = \sqrt{d_p^2 - R^2}$, we can obtain the upper limit of τ :

$$\tau_{\max} = \frac{\sqrt{d_p^2 - R^2} + \sqrt{d_p^2 - R^2 + D^2 - 2D(d_p^2 - R^2)^{1/2} \cos(\theta_p + \arcsin(R/d_p))}}{c} \quad (22)$$

For the same reason, the lower limit of τ is:

$$\tau_{\min} = \frac{\sqrt{d_p^2 - R^2} + \sqrt{d_p^2 - R^2 + D^2 - 2D(d_p^2 - R^2)^{1/2} \cos(\theta_p - \arcsin(R/d_p))}}{c} \quad (23)$$

Therefore the pdf of TOA at BS can be found as:

$$f_{\tau}(\tau) = \begin{cases} \int_{\theta_{d1}}^{\theta_{d2}} \frac{(D^2 - \tau^2 c^2)(D^2 + \tau^2 c^2 - 2D\tau \cos \theta_d)c}{4\pi R^2 (D \cos \theta_d - \tau)^3} d\theta_d & \tau_{\min} \leq \tau \leq \tau_{\max} \\ 0 & \text{otherwise} \end{cases} \quad (24)$$

Using the following transformations:

$$t = \tan \frac{\theta_d}{2}, \cos \theta_d = \frac{1-t^2}{1+t^2}, \sin \theta_d = \frac{2t}{1+t^2}, d\theta_d = \frac{2dt}{1+t^2}$$

The TOA's marginal density explicitly depends on the model parameters of R and D:

$$f_{\tau}(\tau) = \begin{cases} a_1 \arctan\left(\frac{1}{a_4} \tan \frac{\theta_{d2}}{2}\right) + a_2 \frac{\tan \frac{\theta_{d2}}{2}}{(\tan \frac{\theta_{d2}}{2})^2 + a_4^2} + a_3 \frac{\tan \frac{\theta_{d2}}{2}}{((\tan \frac{\theta_{d2}}{2})^2 + a_4^2)^2} - a_1 \arctan\left(\frac{1}{a_4} \tan \frac{\theta_{d1}}{2}\right) \\ - a_2 \frac{\tan \frac{\theta_{d1}}{2}}{(\tan \frac{\theta_{d1}}{2})^2 + a_4^2} - a_3 \frac{\tan \frac{\theta_{d1}}{2}}{((\tan \frac{\theta_{d1}}{2})^2 + a_4^2)^2} & \text{for } \tau_{\min} \leq \tau \leq \tau_{\max} \\ 0 & \text{otherwise} \end{cases} \quad (25)$$

$$\text{Where: } a_4 = \sqrt{\frac{\pi - D}{\pi + D}}, a_1 = \frac{2c\tau^2c^2 - cD^2}{4\pi R^2 \sqrt{\tau^2c^2 - D^2}}, a_2 = \frac{c}{4\pi R^2} \frac{D^2}{(D + \pi)}, a_3 = \frac{(D - \pi)c}{2\pi R^2} \frac{D^2}{(D + \pi)^2}$$

2.4. The Angle of Departure (AOD) pdf at MS

The same equations apply at the mobile when scatter density is referred to the polar coordinate system (r, θ_r) defined at the mobile, the relationship between r and τ is identical in form to the relationship between d and τ , that is:

$$\pi - r = \sqrt{r^2 + D^2 - 2rD \cos(\theta_r)} \quad (26)$$

Squaring both sides of (26) and solving for r results in

$$r = \frac{D^2 - \tau^2c^2}{2(D \cos \theta_r - \pi)} \quad (27)$$

Therefore, the location of the scatterer in the rectangular coordinates is also given by:

$$x_s = r \cos \theta_r = \frac{(D^2 - \tau^2c^2) \cos \theta_r}{2(D \cos \theta_r - \pi)} \quad (28)$$

$$y_s = r \sin \theta_r = \frac{(D^2 - \tau^2c^2) \sin \theta_r}{2(D \cos \theta_r - \pi)} \quad (29)$$

Once the scatterer distribution $f_{x_s, y_s}(x_s, y_s)$ and the relations between (x_s, y_s) and (τ, θ_r) are known, the joint TOA/AOD pdf $f_{\tau, \theta_r}(\tau, \theta_r)$ can be obtained by a Jacobian transformation:

$$f_{\tau, \theta_r}(\tau, \theta_r) = f_{x_s, y_s}(x_s, y_s) \begin{vmatrix} \frac{\partial x_s}{\partial \tau} & \frac{\partial y_s}{\partial \tau} \\ \frac{\partial x_s}{\partial \theta_r} & \frac{\partial y_s}{\partial \theta_r} \end{vmatrix} = \frac{(D^2 - \tau^2c^2)(D^2 + \tau^2c^2 - 2D\pi \cos \theta_r)c}{4\pi R^2 (D \cos \theta_r - \pi)^3} \quad (30)$$

The AOD pdf at the MS could be found by integrate the polar coordinate system representation of the scatterer density function $f_{r, \theta_r}(r, \theta_r)$ with respect to r over the range r_1 to r_2 , where r_1 and r_2 are two pairs of roots for the equations defining the distant scattering discs when the scatterer density is referred to the polar coordinate system (r, θ_r) , in polar coordinates:

$$r^2 + d_r^2 - 2rd_r \cos(\theta_r - \theta_{pM}) \in [0, R^2] \quad (31)$$

Equate (31) to its upper limit of R^2 producing the following two roots

$$r_1, r_2 = \{d_r \cos(\theta_r - \theta_{pM}) \pm \sqrt{R^2 - d_r^2 \sin^2(\theta_r - \theta_{pM})}\} \quad (32)$$

The joint pdf $f_{r, \theta_r}(r, \theta_r)$ can be obtained by another Jacobian transformation with the relations between (x_s, y_s) and (r, θ_r) are known:

$$f_{r, \theta_r}(r, \theta_r) = f_{x_s, y_s}(x_s, y_s) \begin{vmatrix} \frac{\partial x_s}{\partial r} & \frac{\partial y_s}{\partial r} \\ \frac{\partial x_s}{\partial \theta_r} & \frac{\partial y_s}{\partial \theta_r} \end{vmatrix} = r f_{x_s, y_s}(r \cos \theta_r, d \sin \theta_r) \quad (33)$$

$x_s = r \cos \theta_r$
 $y_s = r \sin \theta_r$

Therefore the AOD pdf at MS equals

$$f_{\theta_r}(\theta_r) = \begin{cases} \int_{r_1}^{r_2} f_{r,\theta_r}(r, \theta_r) dr & \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{2d_r \cos(\theta_r - \theta_{pm}) \sqrt{R^2 - d_r^2 \sin^2(\theta_r - \theta_{pm})}}{\pi R^2} & \\ 0 & \text{otherwise} \end{cases} \quad (34)$$

for $\theta_{pm} - \arcsin \frac{R}{d_r} \leq \theta_r \leq \theta_{pm} + \arcsin \frac{R}{d_r}$

3. Results and Analysis

In this section the above theoretical models are validate using simulation, the distance between MS and BS is 2km, and its centre is P(1500,400) m, $\tau_c=2450m$, the joint pdf of TOA/AOA at BS for the uniformly distribution scatterer model is shown in Figure 2 (equation (9)), the radius of cell R is chosen as 200m, equation (13) is plotted in Figure 3, the scatterers are uniformly located within a circles of different radius, equation (25) is plotted in Figure 4, equation (34) is plotted in Figure 5, the pdf derived in this section can be used to simulate a power-delay-angle profile and to quantify angle spread and delay spread^[13] for the given R, D and the centre of the scattering circle.

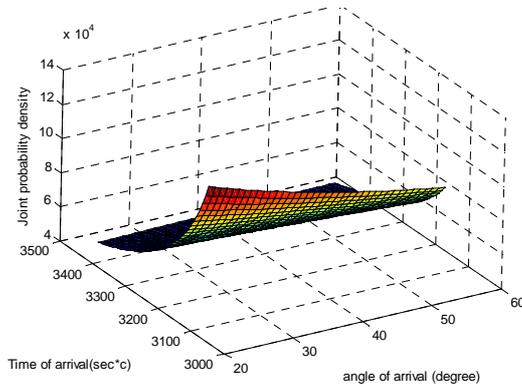


Figure 2. Joint TOA/AOA pdf (P(1500,400); R=200;D=2000; $\tau_c =2450$)

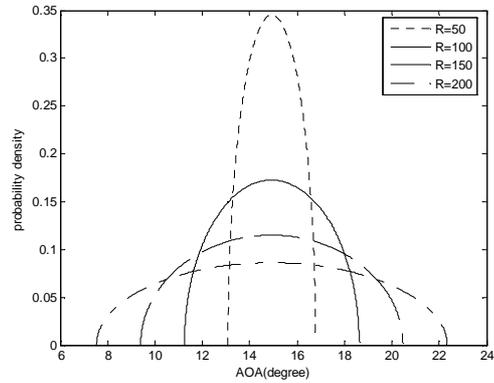


Figure 3. pdf of AOA at BS(P(1500,400); D=2000; $\tau_c =2450$)

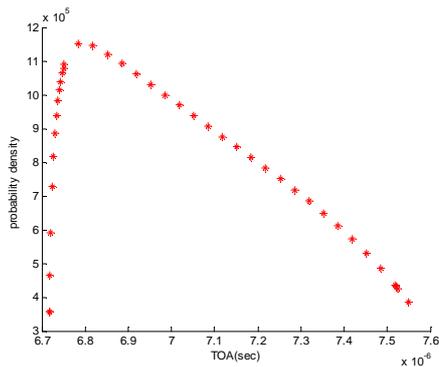


Figure 4. pdf of the TOA at BS(P(1500,300);R=200;D=2000; $\tau_c =2400$)

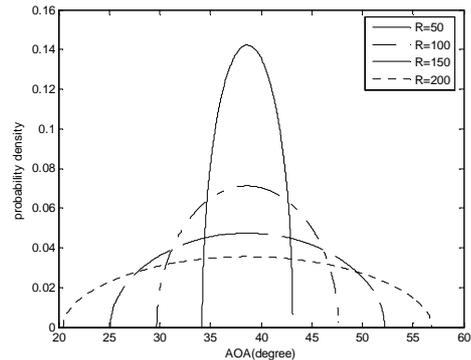


Figure 5. pdf of AOD at MS(P(1500,400); D=2000; $\tau_c =2450$)

4. Conclusion

In this paper, we have derived geometrical channel model assumed that the scatterers are within a circle far from both BS and MS, this type of scattering can occur in hilly and suburban areas due to large scattering structures such as mountains and high building clusters, which have a significant influence on the mobile channel, the joint TOA/AOA and marginal TOA ,AOA and AOD pdf's for the circular scattering model are derived.

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